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DETECTION OF SPRING SUPPORT LOCATIONS IN ELASTIC STRUCTURES USING A GRADIENT-BASED FINITE ELEMENT MODEL UPDATING TECHNIQUE

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A methodology is presented for detection of the location of simple translational spring supports in elastic structures by non-destructive and non-intrusive means, based on measured natural frequencies of the structure. A gradient-based finite element (FE) model updating technique has been used for such detection. Measured modal data along with an initially correlated FE model are used. The present study is a new application of an existing FE model updating technique. The proposed method involves the detection of spring support locations by updating the position parameters of the support in the FE model through the optimization of an error criterion based on the difference between measured and computed natural frequencies of the structural system. The location of the support appears explicitly as an updating parameter in the formulation of the problem. Three different numerical schemes for computation of the gradient of the eigenvalue derivative with respect to the support location have been evaluated. The results and various limitations observed while developing the current method have been presented through numerical examples and a simple experimental example.

1. INTRODUCTION

Many flexible mechanical systems such as fuel pins, heat exchanger tubes, control rods and various instrumented and shrouded tubes used in nuclear power plants and other engineering industries are beam-like components with a number of intermediate supports along their length. In many cases, these intermediate supports are firmly fixed. However, in some cases they may be loosely coupled and may move from their original locations during

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operation say, for example, because of flow-induced vibration. The movement of supports may or may not affect the support stiffnesses depending upon the structural configuration. Undetected, such dislocated supports may deteriorate the system function and consequently jeopardize the safety of the structure *vis-à-vis* the plant safety. Visual inspection of such support locations in the structural system may not be always possible if the structural configuration is complex. Other feasible inspection methods could be exorbitant and time consuming and may lead to extended shutdown of the system *via-à-vis* the plant. One such typical example consists of a number of assemblies of two horizontal coaxial flexible tubes with loosely held spacers to maintain the annular gap between the coaxial tubes, which are generally used in nuclear reactors. These assemblies are required for carrying hot fluid inside the inner tube with an insulating gas-filled annulus between the outer and inner tubes to reduce heat losses. It has been observed over a period of plant operation that the loosely held spacers generally move from their design locations. Locating these spacers appropriately is important for maintaining coaxiality and preventing contact between inner and outer tubes due to bending creep of the inner tube. The conventional method of inspection is expensive and time consuming. Hence, a non-intrusive and non-destructive method for the detection of support locations in the structural system in a quicker but reliable manner is important. A possible method for such a detection could be based on the identification of the spring location using experimental data such as measured natural frequencies and mode shapes of the structural system.

Such an identification of spring locations is similar to the problem of identification of location and extent of damage in structures. Many studies on crack identification have been reported in literature. A review of the research on the damage identification in structures is given by Doebling *et al.* [1]. Most of the methods proposed to detect the location and the size of the damage use change in measured vibration response for the identification. Such an identification requires a mathematical model (e.g., a finite element (FE) model) and experimental modal parameters of the structure. These identification methods are predominantely based on the change in natural frequencies [2–10] or the change in mode shapes [11–19] or dynamically measured flexibility [20–26]. Salawu [27] gives the review of research work on crack identification based on the change in natural frequencies. Another class of damage identification methods, also based on the change in modal parameters, uses a different identification approach which is based on the modification of structural model matrices (such as mass, stiffness and damping matrices) using FE model updating methods [1].

Friswell and Mottershead [28], Imregun and Visser [29] and Mottershead and Friswell [30] give excellent reviews of the model updating methods. The prime purpose of the updating methods is to fine tune the FE model of structures using experimental modal parameters. The FE models require the material properties and physical dimensions of the structural system under consideration. In practice, the construction of the FE models is usually based on a number of simplifying assumptions. Often, such an FE model may not be fully reliable because of the various idealizations made that generally depend upon the understanding and the engineering judgement of individuals involved in the modelling. Thus, any parametric and/or modelling errors or deviations may lead to a model which may not be the true reflection of the "as-built" structure, unless some level of validation is carried out by making use of the measured modal data from experimental tests. The model updating process generally minimizes the uncertainties in the FE model to the extent possible by adjusting/updating the parameters of the model to produce matching test behaviour. The model updating could be done either by direct methods or by sensitivity methods. Using these updating concepts, many studies on the crack identification in structural systems have also been reported [1].

The methods used for crack identification are based either on direct methods or on iterative methods using gradient-based sensitivity information. Direct methods of model updating generally use a closed-form solution by minimizing the modal force error with symmetry of structural matrices as constraints to update the model to produce results matching the experimental modal data. This concept is utilized to compute the model matrices of the damaged structure. Doebling *et al.* [1] give the review of study on crack identification using different direct methods of model updating. These direct methods generally produce exact results matching the experimental modal data. However, the resulting updated FE model may lose physical meaning. Hence, these have not been generally used in practice.

However, the sensitivity methods overcome the limitations of the direct methods but require iterative solution. In the gradient-based sensitivity methods of model updating an important aspect is to define an error function between the computed and the test data. The error could be defined in the modal domain in terms of the difference in measured and computed natural frequencies or both natural frequencies and mode shapes or in frequency domain [28]. Such an error function is usually a highly non-linear function with respect to the updating parameters. The solution for the updating parameters is generally obtained by minimization of the error function through optimization techniques. The iterative solution of the non-linear optimization problem by a gradient search technique requires the formulation and computation of the sensitivity matrix of the error function with respect to the updating parameters. Usually, a local linearization is carried out using a Taylor series expansion by retaining the first order terms to compute the first order sensitivity matrix. In defining the error function as well as in the construction of the sensitivity matrix, the correct pairing up of computed modal data (natural frequencies or both natural frequencies and mode shapes) with the experimental modal data is essential. This is important because the pairing up of computed and test data, based on the sequential order of mode numbers, may not be always correct. This correlation between the computed and the test data is generally established using the modal assurance criterion (MAC) [31]. Another important task in model updating is the selection of the parameters to be updated. The parameters should be chosen with the aim of correcting the recognized uncertainty in the model. Moreover, the computed eigenvalues, eigenvectors, etc., of the FE model should be sensitive to the updating parameters. A number of such sensitivity methods have been discussed by Friswell and Mottershead [28]. Using these methods, many research studies on the crack identification are also reported using residual modal force as an updating parameter. A few examples are the research studies by Chen *et al.* [32], Haug *et al.* [33], Ricles *et al.* [34], Farhat *et al.* [35] and Hemez *et al.* [36].

In this paper, the technique proposed for the detection of support locations is also based on a parameter identification method using gradient-based FE model updating. The application of the updating method is somewhat different in the present study compared to earlier works on damage identification. In many studies on crack identification the concept is based on the use of residual modal dynamics vectors as updating parameters for the estimation of sensitivity matrix and the subsequent updating of system stiffness and mass matrices using modal parameters of the cracked structure to identify the crack. However, explicit use of position vector as an updating parameter for the identification of crack location in the structure is not reported. The problem of detection of support locations is formulated and solved in this paper on the basis of such a concept.

A few studies are also reported on the estimation of optimal support location in mechanical structures to maximize the fundamental frequency. Pitarresi and Kunz [37] have used non-linear least-squares fit of natural frequencies versus support location data in a vibrating plate to optimize the point support locations. Son and Kwak [38] have used a gradient-based optimization method for such a requirement in their study. These concepts are useful for design requirement. However, the objective of the present study is different. The present method explores the possibility of detection of support locations in the existing mechanical structural system based on the change in the first few natural frequencies due to support movement.

In this paper, a problem of two beams with a number of intermediate massless spring supports is considered for simplicity and for the assessment of the proposed technique. It is also assumed that the stiffness of each spring support is an appropriately fixed value that does not change with the location. The solution of this problem has been attempted by updating the position parameters of the support in the FE model using a sensitivity-based model updating method. The location of the support appears explicitly as an updating parameter in the formulation of the problem. The FE model updating method requires an FE model of the structural system for the computation of modal data (i.e., natural frequencies and mode shapes) for given support locations in the structural system. An FE model truly representative of the system with support locations in the design condition is taken as the base-line model. For off-design locations, the proposed technique involves updating the position parameters of the support in the FE model through the optimization of an error function which is defined based on the difference between measured and computed natural frequencies of the structural system. The penalty function method is used for the minimization of the error function.

The theoretical concept, the complete computational implementation used and the various limitations observed during the development of this technique have been presented in this paper. Three different numerical schemes for the computation of the gradient of the eigenvalue derivative with respect to the support location have also been evaluated. The results are presented to validate and assess the current method through a few numerically simulated examples and a simple experimental example.

2. THEORETICAL FORMULATION

As mentioned earlier, a problem of two simple beams with a number of massless intermediate spring supports is used in this study. A schematic diagram of this configuration is shown in Figure 1. It is also assumed that the support springs are shifting along the beam length and this movement does not change their own stiffnesses. The technique proposed in this paper is based on one of the gradient-based model updating methods, i.e., *penalty function method* [28]. The complete formulation used for the specific application of support location is brought out in this section. The error function for this application has been defined by involving natural frequencies only.

Let

- $\{\theta\}^{\mathrm{T}} = [x_1,\ x_2, \ldots, x_l]$ be the vector of the locations of the 1st–lth support in the x direction (see Figure 1), henceforth referred to as the vector of updating parameter.
- $\{\mathbf{Z}_c\}^{\mathrm{T}} = [\lambda_{c1}, \lambda_{c2}, \ldots, \lambda_{cm}]$ be the vector of the eigenvalues (natural frequencies) of the 1st–mth mode computed from FE model for location $[x_1, x_2, \ldots, x_l]$.
- $\{\mathbf{Z}_e\}^{\mathrm{T}} = [\lambda_{e1}, \lambda_{e2}, \ldots, \lambda_{em}]$ be the vector of the eigenvalues of the 1st–mth mode measured experimentally for some support locations which are to be determined.

Let ε be the error function (vector) which is the difference between the measured and the computed natural frequency, i.e.,

$$\{\varepsilon\} = \{\mathbf{Z}_e\} - \{\mathbf{Z}_c\}.$$

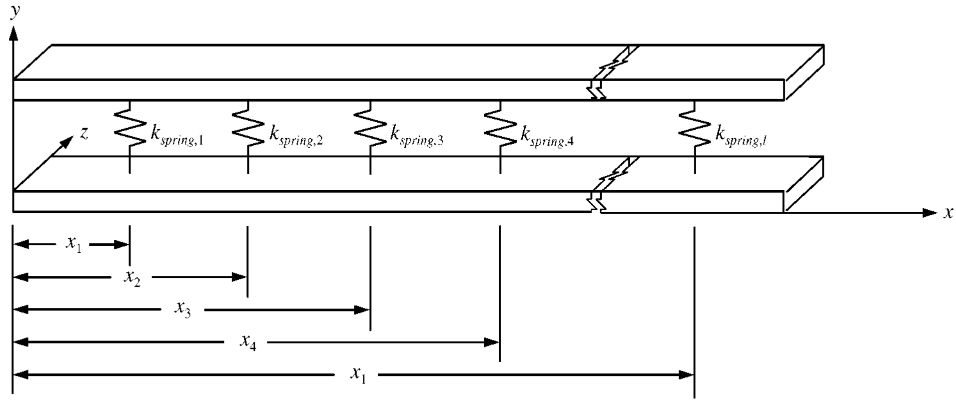

Figure 1. Schematic of two beams with interconnected intermediate supports: ($k_{spring,1}$, $k_{spring,2}$, ... $k_{spring,l}$ = Stiffnesses of 1st to lth supports.

The error vector is a function of the spring support locations vector, i.e. $\{\theta\}$. Taking a first order truncated Taylor series expansion of the error function with respect to the updating parameter vector, the linearized approximation of the error function can be written as

$$\varepsilon = \delta \mathbf{Z} - \mathbf{S}\, \delta\theta, \tag{1}$$

where $\delta\theta$ is the vector of perturbation in the support locations, $\delta\mathbf{Z} = \{\mathbf{Z}_e\} - \{\mathbf{Z}_c\}$ is the eigenvalue error (natural frequency error), and $\mathbf{S} = [\partial\mathbf{Z}/\delta\theta]$ is the sensitivity matrix which is the first derivative of eigenvalues with respect to the support location,

$$\mathbf{S} = \begin{bmatrix} \dfrac{\partial\lambda_{c1}}{\partial x_1} & \dfrac{\partial\lambda_{c1}}{\partial x_2} & \cdots & \dfrac{\partial\lambda_{c1}}{\partial x_l} \\ \dfrac{\partial\lambda_{c2}}{\partial x_1} & \dfrac{\partial\lambda_{c2}}{\partial x_2} & \cdots & \dfrac{\partial\lambda_{c2}}{\partial x_l} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial\lambda_{cm}}{\partial x_1} & \dfrac{\partial\lambda_{cm}}{\partial x_2} & \cdots & \dfrac{\partial\lambda_{cm}}{\partial x_l} \end{bmatrix} \tag{2}$$

The penalty function (**J**) is formed as [28]

$$\mathbf{J}_k\,(\delta\theta) = \varepsilon^{\mathrm{T}}\, \mathbf{W}_\varepsilon\, \varepsilon, \tag{3}$$

where "$\mathbf{W}_\varepsilon$" is the positive diagonal weighting matrix which reflects the confidence level in the frequency measurements. It is generally taken as the reciprocal of the variance (i.e., the squares of the standard deviation) of the corresponding measurements [28].

The vector of desired support locations can be obtained by minimizing '**J**' with respect to '$\delta\theta$' which involves the differentiation of '**J**' with respect to each element of '$\delta\theta$' and setting the result equal to zero. The solution so obtained at each step is, in fact, a weighted non-linear (iterative) least-squares solution. Finally, this leads to the following equation for the vector of support locations after each iteration in the minimization process.

At the kth iteration,

$$\{\boldsymbol{\theta}\}_{k+1} = \{\boldsymbol{\theta}\}_k + [\mathbf{S}^{\mathrm{T}}\mathbf{W}_\varepsilon\mathbf{S}]_k^{-1}\, \mathbf{S}_k^{\mathrm{T}}\, \mathbf{W}_\varepsilon\, \{\mathbf{Z}_e - \mathbf{Z}_c\}_k. \tag{4}$$

The iteration process will continue till the solution converges.

3. COMPUTATIONAL IMPLEMENTATION

To implement the above formulation (equation (4)) for the detection of support locations, the modal data need to be computed for a given $\{\theta\}_k$. In this paper, an FE model is used for this purpose. The next and perhaps the more difficult task is to construct the sensitivity matrix $\mathbf{S}$ from the FE model. In additon, the experimental modal data for some unknown support locations which are to be found, is required to determine $\{\delta\mathbf{Z}\}$ in equation (4).

3.1. ESTIMATION OF SENSITIVITY MATRIX

As seen from equation (2), the construction of the sensitivity matrix, $\mathbf{S}$, requires the derivative of the eigenvalues with respect to the support location (i.e. $\partial\lambda_{ci}/\partial x_j$). This can be obtained by the differentiation of the following characteristic structural dynamic equation:

$$[\mathbf{K} - \lambda\mathbf{M}]\,\{\Phi\} = 0, \tag{5}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices and $\{\Phi\}$ is the normalized eigenvector of the structural system and λ is the eigenvalue.

Mathematically, each element of the sensitivity matrix at the kth iteration can be written as [28]

$$\mathbf{S}_{ij,k} = \left(\frac{\partial\lambda_{ci}}{\partial x_j}\right)_k = \{\Phi_i\}_k^{\mathrm{T}}\left[\frac{\partial\mathbf{K}}{\partial x_j} - \lambda_{ci}\,\frac{\partial\mathbf{M}}{\partial x_j}\right]_k \{\Phi_i\}_k, \tag{6}$$

where $\lambda_{ci,k}$ and $\{\Phi_i\}_k$ are the ith eigenvalue and eigenvector of the structural system and x_j is the jth support location, all at the kth iteration.

In general, when the stiffness and mass matrices of the structural system are continuous functions of the updating parameters, equation (6) can be directly used to construct the sensitivity matrix. This is so, for example, when the parameters are physical dimensions, material properties and boundary stiffnesses of the structural system. For example, if the updating parameter is modulus of elasticity (E_j) of the jth element of the FE mesh, the analytical differentiation of eigenvalue ($\partial\lambda_{ci}/\partial E_j$) using equation (6) is possible. Such updating parameters, in fact, generally change the value of coefficients at fixed locations in the system stiffness and mass matrices during the after model updating process. Many updating problems studied earlier in the literature are of this type, for example, the various works reported earlier [28].

However, in the present problem the updating parameter is the support position (x) and not the value of the spring support stiffness (which is an appropriately fixed value). The shift in support location is reflected as a change in the nodes to which the springs are attached. This is further reflected in the FE model in the form of discrete (step) jumps in the value of those elements of the stiffness matrix which correspond to the direct stiffness at the nodes to which the springs are attached. In short, this is equivalent to a change in the model itself, for each set of support locations. Thus, it is not possible to compute the sensitivity matrix analytically using equation (6) in the present case.

Thus, an alternate method has to be used to derive the eigenvalue derivative for the present problem. This has been done in three ways — the first two are based on equation (6) and the third one is based on the basic definition of derivative ($\partial\lambda_{ci}/\partial x_j$). In the case of detection of support locations, the derivative ($\partial\mathbf{M}/\partial x$) is zero since it is assumed that the mass of the spring-loaded supports is negligible and hence, the change of support locations would not change the overall mass matrix. The derivative ($\partial\mathbf{K}/\partial x$) can be estimated by the following relations [39].

(i) *Eigenvalue Derivative* 1 (Gradient estimation based on local linearization).

The stiffness derivative with respect to the jth support location is

$$\left[\frac{\partial \mathbf{K}}{\partial x_j}\right]_k = \frac{[\mathbf{K}_k - \mathbf{K}_k^{j*}]}{(x_{j,k} - x_{j,k}^*)},$$

where $x_{j,k}$ is the location of the jth support in the system FE model at the kth iteration, $x_{j,k}^*$ is the nodal position in the system FE model adjacent to the nodal position of the jth support (i.e., $x_{j,k}$) at the kth iteration, $\mathbf{K}_k$ is the stiffness matrix of the structural system at the kth iteration with spring supports at $\{\theta\}_k$, and $\mathbf{K}_k^{j*}$ is the stiffness matrix of the structural system with the jth support at location, $x_{j,k}^*$, and the other supports remaining the same as that at the kth iteration.

Substituting this derivative into equation (6), each element of the sensitivity matrix can be written as

$$\mathbf{S}_{ij,k} = \left(\frac{\partial \lambda_{ci}}{\partial x_j}\right)_k = \{\Phi_i\}_k^T \frac{[\mathbf{K}_k - \mathbf{K}_k^{j*}]}{(x_{j,k} - x_{j,k}^*)} \{\Phi_i\}_k. \tag{7}$$

(ii) *Eigenvalue Derivative* 2 (Gradient estimation based on two successive iterations).

Based on a concept similar to Ricles and Kosmatka [34] in their study on crack identification in structures, a modified estimate of the system stiffness derivative has been constructed as given below

$$\left[\frac{\partial \mathbf{K}}{\partial x_j}\right]_k = \frac{[\mathbf{K}_k^{j} - \mathbf{K}_{k-1}]}{(x_{j,k} - x_{j,k-1})},$$

where $x_{j,k}$ is the location of the jth support in the system FE at the kth iteration, $x_{j,k-1}$ is the location of the jth support in the system FE at the $(k-1)$th iteration, $\mathbf{K}_{k-1}$ is the stiffness matrix of the structural system at the $(k-1)$th iteration, and $\mathbf{K}_k^j$ is the stiffness matrix of the structural system with the jth support at location, $x_{j,k}$, and the other supports remaining the same as that at the $(k-1)$th iteration.

Substituting this derivative into equation (6), each element of the sensitivity matrix can be written as

$$\mathbf{S}_{ij,k} = \left(\frac{\partial \lambda_{ci}}{\partial x_j}\right)_k = \{\Phi_i\}_k^T \frac{[\mathbf{K}_k^{j} - \mathbf{K}_{k-1}]}{(x_{j,k} - x_{j,k-1})} \{\Phi_i\}_k\,. \tag{8}$$

(iii) *Eigenvalue Derivative* 3 (Direct gradient estimation based on basic definition).

$$\mathbf{S}_{ij,k} = \left(\frac{\partial \lambda_{ci}}{\partial x_j}\right)_k = \frac{\lambda_{ci,k-1}^* - \lambda_{ci,k-1}}{x_{j,k} - x_{j,k-1}}, \tag{9}$$

where $x_{j,k}$ and $x_{j,k-1}$ are as defined in Eigenvalue Derivative 2 above, $\lambda_{ci,k-l}$ is the ith eigenvalue at the $(k-1)$th iteration and $\lambda_{ci,k}^*$ is the ith eigenvalue at the kth iteration for the jth support location, $x_{j,k}$, such that the other support locations remain the same as that at the $(k-1)$th iteration. This is nothing but the eigenvalue for incremental change in the jth updating parameter.

A similar concept has been used earlier for eigenvector derivatives for continuous updating parameter problem by Suther *et al.* [40].

The computational effort involved in the estimation of eigenvalue derivative is least for Eigenvalue Derivative 1 (equation (7)) and more for Eigenvalue Derivative 3 (equation (9)). The usefulness, effectiveness and limitations of these formulations for eigenvalue derivative estimations is not known and reported in th literature. This exercise was also carried out in the way of development of the proposed method.

3.2. COMPARISON OF COMPUTED AND MEASURED MODAL DATA

As mentioned earlier, the computed eigenvalues $\{\mathbf{Z}_c\}$ for a set of support locations must be paired correctly with the measured eigenvalues $\{\mathbf{Z}_e\}$, if one is using gradient-based model updating method. The modal assurance criteria (MAC) [31] are used for this purpose in this study. The frequencies of mode pairs with MAC $\geq 0{\cdot}5$ have been used in all the numerical simulations presented in section 4.

3.3. ITERATIVE PROCESS

The iterative process in equation (4) starts with an initial guess for the support locations $\{\theta\}_0$. However, for the construction of sensitivity matrix, two locations of each support are required. To start the iteration, the second set of location was assumed to be the nodes adjacent to nodal locations of the initial guess vector. Using each of the three equations (7)–(9) for the eigenvalue derivatives the sensitivity matrix was constructed as in equation (2) and the iteration was carried out as per equation (4). The computed locations at each iteration by equation (4) may not coincide with the nodes of FE model. One way to proceed further is that the FE model of the structural system be re-discretized in the vicinity of the updated/computed location of the supports at every iteration, if exact location is required. But this practice would definitely be time consuming and would require substantial computational effort for a large-sized problem. Alternatively, one can assumed the updated locations of the support to the nearest nodes at the end of each iteration if the accuracy requirement is not very strict, i.e. if it is sufficient to obtain the support locations in terms of the nearest node. The latter scheme was adopted for the study reported. The advantage of this scheme is that the matrices of the structural system do not have to be generated at each iteration. Only the stiffness of the spring supports has to be placed at the correct nodes in the global stiffness matrix. The only limitation of this scheme is that if the actual location of the support is between two nodes of the FE model, it can only approximate this location to the nearest node. Hence, a very fine FE mesh is required to avoid large errors in the detection of support locations if high accuracy is required.

After each iteration, the eigenvalues and eigenvectors were computed using the updated FE model for the new set of support locations. The computed eigenvalues were then compared with the measured natural frequencies and progressively mode pairs with MAC $\geqslant 0{\cdot}5$ are also included in the next iteration. This iterative process continues till the problem converges. In all the examples, the following convergence criteria were used:

$$\partial Z_{i,k} \leqslant 10^{-5} \quad \text{and} \quad \mathrm{MAC}_{i,k} \geqslant 0{\cdot}9, \quad i = 1, 2, 3, \ldots, m.$$

4. ASSESSMENT AND VALIDATION

To proceed to the detection of spring support locations in the structural system from different initial guesses using the proposed method, a knowledge of the following modal parameters is essential.

(a) *Measured modal parameters.* It is a set of measured natural frequencies and mode shapes of the structure corresponding to some support locations which are to be determined by the proposed method. This set of data is henceforth referred to as the target data for iterative solution. Any of the all-known experimental techniques for modal analysis [41] can be used to obtain these modal parameters. In the present numerical study, analytically simulated modal data are used in place of the experimental modal data for assessment of the method.

(b) *Initial guess for support locations in the FE model.* The initial guess data are also a set of natural frequencies and mode shapes (i.e., eigenvalues and eigenvectors) computed from the FE model corresponding to initially assumed support location in the structure.

An assessment of the effectiveness of the proposed method and the limitations of the different ways of estimation of the eigenvalue derivative have been brought out through numerical examples. In all the numerical examples, beams of two different dimensions and spring supports of stiffness $1{\cdot}0 \times 10^4$ N/m have been chosen for the present study. Dimensions (length × depth × width) of both the beams used for the study are (2000 mm × 50 mm × 25 mm) and (2000 mm × 25 mm × 12·5 mm) respectively. The material properties Young's modulus of elasticity (E) and density (ρ) for both the beams are chosen to be 70×10^9 N/m^2 and 2666·67 kg/m^3 respectively. In all the examples considered, the FE model was constructed using simple beam elements and a massless spring element for the spring support. A two-noded beam element with each node restricted to two degrees of freedom (2 d.o.f.—one for bending displacement and the other for bending rotation) is used.

To check the computational implementation of the proposed technique and the three different eigenvalue derivatives, a few examples (from simple to complex) were solved before solving the problem defined in section 1. Here, only the first three modes have been used for the support location detection. However, depending upon the structural configuration and change in their natural frequencies due to shift to supports, a few higher modes can be included in the detection process. The results and various limitations observed for different eigenvalue derivatives are discussed in subsequent sub-sections.

4.1. RESULTS USING EIGENVALUE DERIVATIVE 1

To start with a simple problem of a single beam with a single spring support was solved to check the computational implementation. The schematic of the FE model of a cantilever beam with a spring support is shown in Figure 2(a). The FE model consists of 19 beams elements (i.e., 20 nodes). In the target data, it was assumed that the translational spring support was at node location, $x_1 = 1473{\cdot}7$ mm (node 15) from the fixed end of the beam. The computed first three natural frequencies of the beam from the FE model for this target location of spring are 9·373, 32·355 and 90·410 Hz respectively. These frequencies are now considered as target data.

The following two initial guesses for the spring locations in the FE model were used to detect the target support location through the proposed method using a support position vector x_1 as updating parameter. The sensitivity matrix was constructed using Eigenvalue Derivative 1.

Initial guess 1: $x_1 = 947{\cdot}37$ mm (node 10)
Natural frequencies—6·331, 33·471 and 90·117 Hz.

Initial guess 2: $x_1 = 631{\cdot}58$ mm (node 07)
Natural frequencies—5·468, 32·976 and 90·559 Hz.

As can be seen from the values of frequencies, the first natural frequency for the initial guess of support location in the beam significantly varies from target and the difference in the other two natural frequencies is small. Based on MAC value ($\geqslant 0{\cdot}5$), $\delta\mathbf{Z}$ was defined using these difference in the natural frequencies and the iterative solution of equation (4) was applied for the detection of target support location. It was observed that the target support location could be detected at the first and third iterations from initial

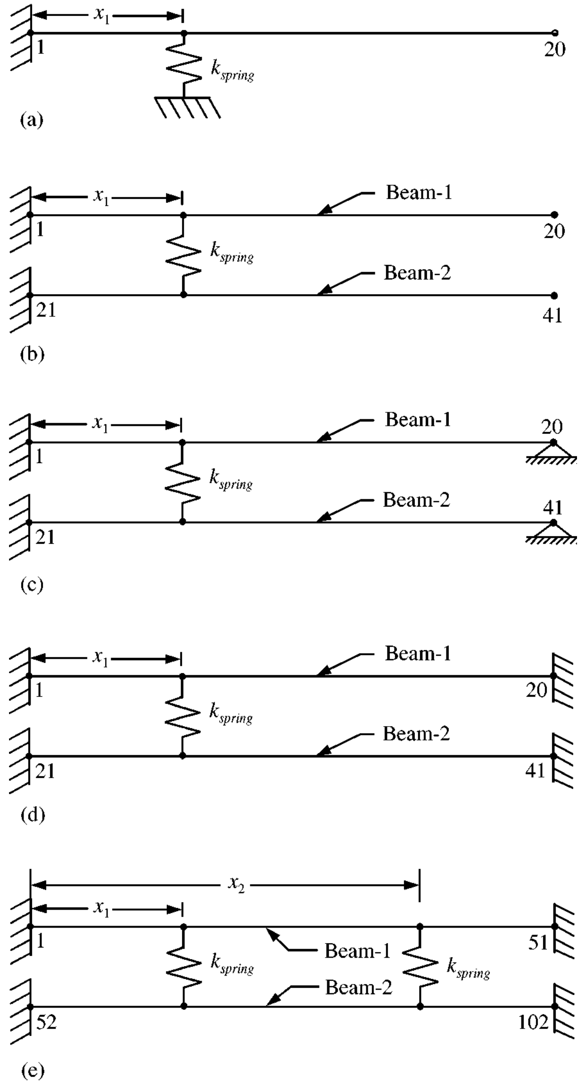

Figure 2. Beam models used in the numerical simulations: (a) A Cantilever Beam with a Spring Support; (b) Two Cantilever Beams with a Spring Support; (c) Two Fixed–Simply Supported Beams with a Spring Support; (d) Two-Fixed–Fixed Beams with a Spring Support; (e) Two Fixed–Fixed Beams with Two Spring Supports; k_{spring} = 1E04 N/m; •, Node point.

guesses 1 and 2, respectively, without any error. This is expected for simulated example without noise.

A few more exercises were also carried out for different initial guesses or/and the target frequencies for different target support locations. It was observed that the problem did not converge towards the desired target in many cases. One such non-convergence iteration history is shown in Figure 3 for case 1a of Table 1. Thus, it was concluded that the construction of sensitivity matrix using Eigenvalue Derivative 1 (equation (7)) based on local gradient concept may not be always effective. Hence, the Eigenvalue Derivative 2 was used for the construction of sensitivity matrix for further exercises as discussed below.

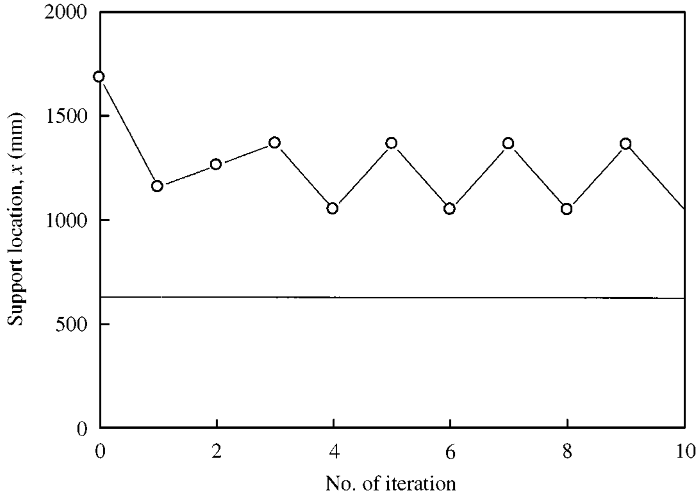

Figure 3. Iteration history of case 1a by using Eigenvalue Derivative 1: —○—, iterative process; ——, target location.

TABLE 1

Results using Eigenvalue Derivative-2 for the example shown in Figure 2(a)

		Case 1a		Case 2a		Case 3a	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	1684·21 (node 17)†	631·58 (node 07)	2000·00 (node 20)	631·58 (node 07)	2000·00 (node 20)	210·53 (node 03)
Natural frequency (Hz)	1st	10·881	5·468	12·394	5·468	12·394	5·170
	2nd	32·427	32·976	34·747	32·976	34·747	32·290
	3rd	90·118	90·559	90·925	90·559	90·925	90·142
No. of iterations to achieve target		04		10		19	

† Beam 1 node number for spring support position.

4.2. RESULTS USING EIGENVALUE DERIVATIVE 2

The detection of spring support for the case 1a of Table 1 was again carried out but using the sensitivity matrix based on Eigenvalue Derivative 2 (equation (8)) and the support position vector x_1 as updating parameter. It was observed that solution converged to the target at the fourth iteration (see Table 1). It was also observed that the solution converged to the target without any error for the simulated example showed in Figure 2(a) in all cases. Results are listed in Table 1.

Further, the exercise of spring support detection was carried out for the example of two cantilever beams with a single spring support as shown in Figure 2(b). It was observed that some of the target locations could not be detected even when a number of different initial guesses were used. One such cases for the example shown in Figure 2(b) was found, wherein the location of the target data of case 1b in Table 2 was not detected from the initial guess.

TABLE 2

Results using Eigenvalue Derivative-3 for the example shown in Figure 2(b)

		Case 1b		Case 2b		Case 3b	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	1789·47 (nodes 18–38)†	947·37 (nodes 10–30)	1789·50 (nodes 18–38)	210·53 (nodes 03–23)	210·53 (nodes 03–23)	1473·68 (nodes 15–35)
Natural frequency (Hz)	1st	4·741	4·427	4·741	2·619	2·619	4·733
	2nd	13·733	6·490	13·733	5·170	5·170	14·386
	3rd	26·459	23·415	26·459	16·314	16·314	18·567
No. of iterations to achieve target		12		03		04	

† Beam 1–2 node numbers.

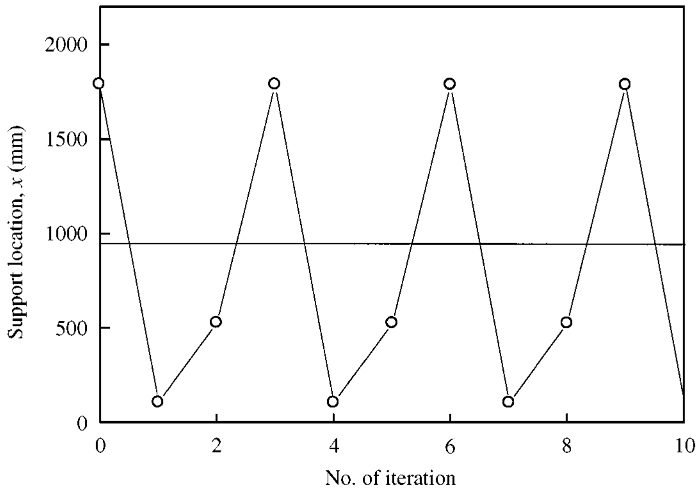

Figure 4. Iteration history of case 1b by using Eigenvalue Derivative 2: —○—, iterative process; ——, target location.

The iteration history of this non-convergence is graphically shown in Figure 4. Thus, the use of Eigenvalue Derivative 2 (equation (8)) is also not a fully reliable solution for the present requirement of detection of support locations in elastic structures.

4.3. RESULTS USING EIGENVALUE DERIVATIVE 3

To start with, the solution of the case 1b (unsolved case using Eigenvalue Derivative 2) in Table 2 was attempted first. It was observed that the solution converged to the target at the 12th iteration (see Table 2). In fact, the solution for the example shown in Figure 2(b) converged to the target in all simulated cases using Eigenvalue Derivative 3 without any error. Results are listed in Table 2.

A few more exercises were repeated for the assembly of two beams with a single intermediate spring support having different boundary conditions as shown in Figure 2(c)

TABLE 3

Results using Eigenvalue Derivative-3 for the example shown in Figure 2(c)

		Case 1c		Case 2c		Case 3c	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	842·10 (nodes 09–29)†	631·58 (nodes 07–27)	842·10 (nodes 09–29)	210·53 (nodes 03–23)	947·37 (nodes 10–30)	210·53 (nodes 03–23)
Natural frequency (Hz)	1st	17·013	14·642	17·013	11·475	17·927	11·475
	2nd	24·300	23·257	24·300	22·694	25·203	22·694
	3rd	40·178	41·339	40·178	37·087	38·451	37·087
No. of iterations to achieve target		03		02		03	

† Beam 1–2 node numbers.

TABLE 4

Results using Eigenvalue Derivative-3 for the example shown in Figure 2(d)

		Case 1d		Case 2d		Case 3d	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	842·10 (nodes 09–29)†	421·05 (nodes 05–25)	842·10 (nodes 09–29)	210·53 (nodes 03–23)	947·37 (nodes 10–30)	526·32 (nodes 06–26)
Natural frequency (Hz)	1st	23·265	18·053	23·265	16·636	23·962	19·264
	2nd	34·504	33·105	34·504	32·921	34·967	33·291
	3rd	46·839	47·858	46·839	45·736	45·537	48·928
No. of iterations to achieve target		06		03		05	

† Beam 1–2 node numbers.

and 2(d). In all the cited numerical examples, it was observed that the spring support was detected reliably without any error. The results are listed in Tables 3 and 4. In fact, all the unsolved cases found while attempting to use Eigenvalue Derivative 1 and 2 were successfully solved using Eigenvalue Derivative 3. Thus, the use of Eigenvalue Derivative 3 (equation (9)) based on basic definition of differentiation for the construction of sensitivity matrix is shown to be reliable for the detection of the support. Hence, Eigenvalue Derivative 3 was used for solving the problem of multiple supports between two fixed beams as defined in section 1.

4.4. DETECTION OF TWO SPRING SUPPORTS BETWEEN TWO FIXED–FIXED BEAMS

The schematic of the FE model of this structural configuration is shown in Figure 2(e). Here, the support locations, x_1 and x_2, for both the spring supports are chosen as the updating parameters simultaneously. Many exercises have been carried out to determine

TABLE 5

Results of the study for unsymmetrically placed two spring supports

		Case 1e		Case 2e		Case 3e	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	520·0 (nodes 14–65)†	440·0 (nodes 12–63)	520·0 (nodes 14–65)	440·0 (nodes 12–63)	520·0 (nodes 14–65)	440·0 (nodes 12–63)
	x_2 (mm)	1880·0 (nodes 48–99)	1720·0 (nodes 44–95)	1880·0 (nodes 48–99)	1400·0 (nodes 36–89)	1880·0 (nodes 48–99)	1200·0 (nodes 31–82)
Natural frequency (Hz)	1st	19·217	18·769	19·217	22·088	19·217	23·341
	2nd	33·295	33·246	33·295	34·101	33·295	35·226
	3rd	48·950	48·984	48·950	51·479	48·950	49·536
No. of iterations to achieve target		05		06		09	

† Beam 1–2 node numbers.

the location of both the spring supports for the different target data from the same initial guess for different cases. These are successfully detected as can be seen from the details of the results reported in Tables 5 and 6. The iteration history for the convergence of the problems for cases 1e and 9e is shown in Figure 5 for two different cases.

Defining the square root of the summation of the square of the percentage frequency errors (SRSS of Freq. Errors) at the kth iteration as

$$\text{SRSS of Freq. Errors} = \left(\sum_{i=1}^{m} E_{i,k}^2 \right)^{1/2},$$

where $E_{i,k} = ((fn_{ci} - fn_{ei})/fn_{ei})_k \times 100\%$ is the percentage frequency error at the kth iteration and fn_{ci} and fn_{ei} are the ith computed and measured natural frequency of the structural system respectively.

The graphical representation of the SRSS of Freq. Errors of all the three modes with iteration numbers is shown in Figure 6. This figure also clearly indicates the convergence towards the target.

The proposed method using Eigenvalue Derivative 3 has further been applied and tested on a simple laboratory experimental set-up of two parallel tubes, which is as discussed in section 5.

5. EXPERIMENTAL EXAMPLE

A laboratory-scale experiment consists of two parallel tubes made of steel which are inter-connected by a rubber band [39, 42]. The schematic of the set-up is shown in Figure 7. The details of the dimensions and the boundary conditions of both the tubes are also marked in the figure. A modal test was carried out using impulse-response method [41]. It was assumed that the spring action of the rubber band was linear for the small levels of excitation used in the test. Modal tests were conducted for two different locations of the rubber band (656·5 and 746·5 mm from one end), and the measured natural frequencies of the set-up are listed in Table 7. The FE model of the set-up was constructed using beam

TABLE 6

Results of the study for symmetrically placed two spring supports

		Case 4e		Case 5e		Case 6e	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	520·0 (nodes 14–65)†	240·0 (nodes 07–58)	520·0 (nodes 14–65)	280·0 (nodes 08–59)	520·0 (nodes 14–65)	360·0 (nodes 10–61)
	x_2 (mm)	1880·0 (nodes 48–99)	1760·0 (nodes 45–96)	1880·0 (nodes 48–99)	1720·0 (nodes 44–95)	1880·0 (nodes 48–99)	1640·0 (nodes 42–93)
Natural frequency (Hz)	1st	19·217	17·041	19·217	17·428	19·217	18·565
	2nd	33·295	32·986	33·295	33·040	33·295	33·221
	3rd	48·950	46·542	48·950	47·211	48·950	48·885
No. of iterations to achieve target		08		15		05	

		Case 7e		Case 8e		Case 9e	
Parameters		Initial guess	Target data	Initial guess	Target data	Initial guess	Target data
Spring support location	x_1 (mm)	520·0 (nodes 14–65)†	440·0 (nodes 12–63)	520·0 (nodes 14–65)	600·0 (nodes 16–67)	520·0 (nodes 14–65)	800·0 (nodes 21–72)
	x_2 (mm)	1880·0 (nodes 48–99)	1560·0 (nodes 40–91)	1880·0 (nodes 48–99)	1400·0 (nodes 36–87)	1880·0 (nodes 48–99)	1200·0 (nodes 31–82)
Natural frequency (Hz)	1st	19·217	20·137	19·217	23·851	19·217	26·696
	2nd	33·295	33·542	33·295	34·930	33·295	37·780
	3rd	48·950	50·627	48·950	52·142	48·950	48·643
No. of iterations to achieve target		09		07		14	

† Beam 1–2 node numbers.

elements for both the tubes and a spring element for the rubber band. Sinha [39] gave a more detailed description of the experiment and FE modelling.

Once again, the detection of the spring location has been carried out using the position of the spring, x_1, as the updating parameter and the sensitivity matrix using Eigenvalue Derivative 3. As shown in Table 7, the target location for both cases has been detected at the third and fourth iterations from the initial guess of the rubber spring at 508·44 mm. The position of the spring for both cases has a very small error of the order of 0·0366 and -0.672% from their target locations respectively. The detection would have been even better by using finer elements in the FE model. Hence, the Eigenvalue Derivatives 3 is once again found to be reliable even for the experimental example which is expected to have some noise contamination in the measured data.

The usefulness of different Eigenvalue Derivatives is quickly summarized in Table 8. Although one or two supports of same stiffness and beam elements have been used for the development of the method, these are not the constraints of the method. The method can be used even when the support stiffness values are different. Similarly, the method can also be

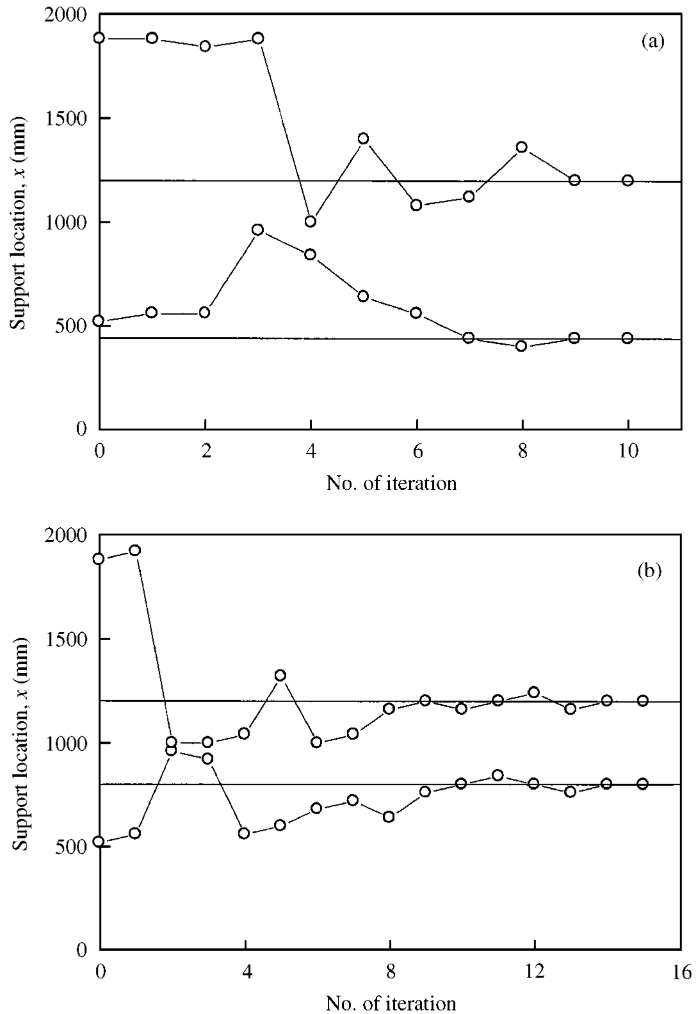

Figure 5. Iteration history for support locations detection by using Eigenvalue Derivative 3: (a) Case 1e: Unsymmetrically placed two supports; (b) Case 9e: Symmetrically placed two supports; —○—, iterative process; ——, target location.

applied easily to more than two supports and plate structures. In fact, this method was used for solving a typical problem encountered in nuclear power plants [39, 42].

6. CONCLUDING REMARKS

A new method for the detection of locations of interconnecting intermediate spring supports in beam structures as a solution of an inverse vibration problem has been presented. The proposed method uses the natural frequencies of the structural system for the detection of such support locations. The methodology presented makes use of a baseline FE model along with the measured modal data in an iterative gradient-based model updating techniques. This is a unique kind of application in the domain of model updating techniques. The validation and advantages of the proposed method have been presented

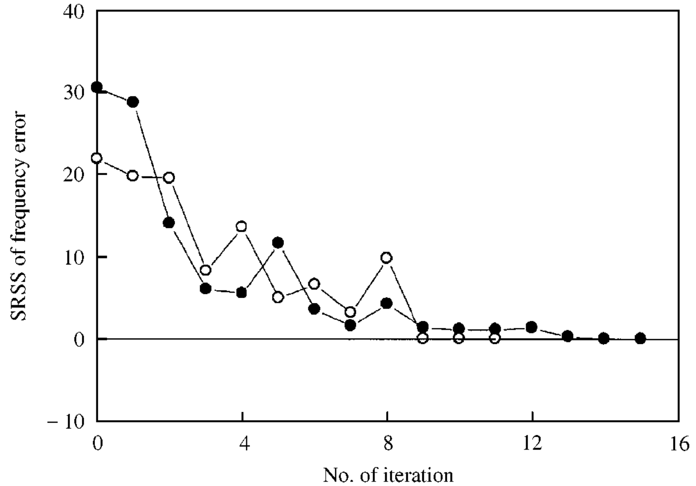

Figure 6. Convergence trend of the error in the natural frequencies with iteration number of cases 1e and 9e: —●—, symmetrically placed supports; —○—, unsymmetrically placed supports; ——, target.

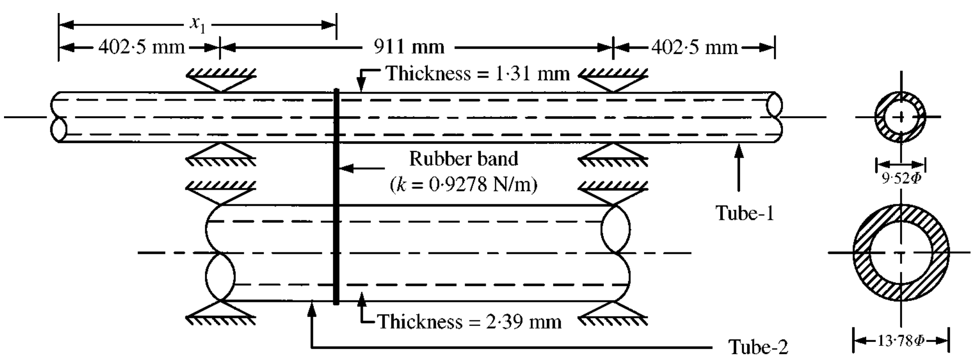

Figure 7. Laboratory experimental set-up.

TABLE 7

Results of spring location for the experimental example

Parameters		Case 1 Initial value	Case 1 Target (test) data	Case 2 Initial value	Case 2 Target (test) data
Spring support location	x_1 (mm)	508·44 (nodes 25–88)†	656·50 (nodes 32–95)	508·44 (nodes 25–88)	746·50 (nodes 36–99)
Natural frequency (Hz)	1st	18·0531	20·938	18·0531	21·875
	2nd	29·081	28·750	29·081	29·375
	3rd	37·219	39·375	37·219	39·375
Using Eigenvalue Derivative 3	Estimated location (% error)	656·74 mm (+ 0·036)		741·48 mm (− 0·672)	
	No.of iterations required	03		04	

† Tube 1–2 node numbers.

TABLE 8

Summary of usefulness of different eigenvalue derivatives

Examples	Description of example	Sensitivity matrix	Results	Remarks
Figure 2(a)	A cantilever beam with a spring support	Using Eigenvalue Derivative 1	Detected target support location for few cases	Unrealiable for most cases (see Figure 3 for case 1a of Table 1)
		Using Eigenvalue Derivative 2	Detected target support location for all simulated cases (Table 1)	Reliable solution without any error
Figure 2(b)	Two cantilever beams with a spring support	Using Eigenvalue Derivative 2	Not able to detect target support location	Not useful (see Figure 4 for case 1b of Table 2).
		Using Eigenvalue Derivative 3	Detected target support location for all simulated cases (Table 2)	Reliable solution without any error
Figure 2(c)	Two fixed–simply supported beams with a spring support	Using Eigenvalue Derivative 3	Detected target support locations for all simulated cases. See Tables 3–6	Reliable solution without any error. (see Figures 5 and 6 for cases 1e and 9e)
Figure 2(d)	Two fixed–fixed beams with a spring support			
Figure 2(e)	Two fixed–fixed beams with two spring supports			
Figure 7 (experimental example)	Two parallel tubes with an interconnection by a rubber band spring	Using Eigenvalue Derivative 3	Detected the target location of rubber band spring for two cases reliably within an error of 1%	Error can be reduced by use of finer FE mesh

through a few numerical simulations and a simple experimental example. It is shown that the support locations could be detected reliably and in a small number of iterations for the example problems considered.

It is important to note that in the analysis presented here the support locations appear directly in the problem considered here as updating parameters. Therefore, the system mass and stiffness matrices are not a continuous function of the vector of updating parameters. The shift in support locations is reflected as a change in the nodes to which the springs are attached. This is further reflected in the FE model in the form of discrete (step) jumps in the value of those elements of the stiffness matrixs which correspond to the direct stiffness at the nodes to which the springs are attached. Thus, it is not possible to estimate the eigenvalue derivative analytically for the construction of the sensitivity matrix. Hence, three possible mathematical alternatives for the eigenvalue derivative estimation have been defined and evaluated in the present study. It was observed that Eigenvalue Derivative 3 based on the basic definition work best for the detection of the support locations, although it is slightly more time consuming compared to the other two alternatives. The method, based on this derivative, is shown to be reliable.

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